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Time periodic flows of an incompressible viscous fluid in perturbed channels

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1 The time periodic Poiseuille flow

In this section, for a straight channel in $\mathbb{R}^n (n = 2, 3)$, which is parallel to the x_1-axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the x_1-axis.

In the case $n = 2$, for $a > 0$ we suppose $\Sigma := (-a, a)$. In the case $n = 3$, we suppose that Σ is a bounded smooth simply connected domain in $\mathbb{R}^2$. We write

$$\omega = \mathbb{R} \times \Sigma.$$

Σ is a cross section of the channel ω.

In ω, we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \mathbf{0} \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R} \times \omega, \tag{1.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \mathbb{R} \times \partial\omega \tag{1.3}$$

with the time periodic condition and the flux condition

$$\boldsymbol{u}(t) = \boldsymbol{u}(t + T) \quad \text{in} \quad \omega \tag{1.4}$$

$$\int_\Sigma \boldsymbol{u}(t) \cdot \boldsymbol{n} dS = \alpha(t) \quad (t \in \mathbb{R}), \tag{1.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of the fluid motion in ω, respectively, ν is the given viscosity constant, $T(> 0)$ is a given constant, $\boldsymbol{n}$ is the unit parallel vector to the x_1-axis and $\alpha(t)$ is a given T-periodic real function.

Since we look for a solution pallalel to the x_1-axis, we may assume that

$$\boldsymbol{u}(t, x) = (v(t, x), 0) \qquad (n = 2),$$
$$\boldsymbol{u}(t, x) = (v(t, x), 0, 0) \quad (n = 3).$$

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Then it follows that v does not depend on x_1 from (1.2), $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0}$ and p depends only on t and x_1 from (1.1). Therefore we obtain the equation

$$\frac{\partial v}{\partial t} - \nu\Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in} \quad \mathbb{R}\times\Sigma, \tag{1.6}$$

where $\Delta = \partial^2/\partial x_2^2$ $(n=2)$, $\Delta = \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ $(n=3)$. It is easy to see that v does not depend on x_1 and p depends only on t and x_1. Therefore it follows from the equation (1.6) that $\partial v/\partial t - \nu\Delta v$ and $\partial p/\partial x_1$ depends only on t. Integrating (1.6) on Σ, we obtain

$$p(t,x_1) = -\frac{1}{|\Sigma|}\left(\alpha'(t) - \nu\int_\Sigma \Delta v(t)dS\right),$$

where $|\Sigma|$ is the Lebesgue measure of Σ. Therefore there exists a time periodic solution $\boldsymbol{u}$ of the Navier-Stokes equations (1.1)–(1.5) in ω, with the form $\boldsymbol{u} = (v,0)$ or $\boldsymbol{u} = (v,0,0)$, if and only if v is a solution of the problem

$$v' + \nu Av - \frac{\nu}{|\Sigma|}(Av,e)e = \frac{\alpha'}{|\Sigma|}e \tag{1.7}$$

with the time periodic condition and the flux condition

$$v(t) = v(t+T) \quad (t\in\mathbb{R}), \tag{1.8}$$

$$(v(t),e) = \alpha(t) \qquad (t\in\mathbb{R}), \tag{1.9}$$

where $e(y) = 1$ $(y\in\Sigma)$, $A = -\Delta$ with the domain $D(A) = H^2(\Sigma)\cap H_0^1(\Sigma)$, $(v,e) = \int_\Sigma ve dS$.

Before stating the time periodic result, we introduce the function space. Let X be a Banach space. We set

$$\begin{aligned} H_\pi^1(\mathbb{R}) &= \{\varphi\in H_{\rm loc}^1(\mathbb{R});\varphi(t) = \varphi(t+T) \text{ a.e. } t\in\mathbb{R}\},\\ L_\pi^2(\mathbb{R};X) &= \{\varphi\in L_{\rm loc}^2(\mathbb{R};X);\varphi(t) = \varphi(t+T) \text{ in } X \text{ for a.e. } t\in\mathbb{R}\},\\ C_\pi(\mathbb{R};X) &= \{\varphi\in C(\mathbb{R};X);\varphi(t) = \varphi(t+T) \text{ in } X \text{ for } t\in\mathbb{R}\}. \end{aligned}$$

Beirão da Veiga [4] proved that for $n\geq 2$ if a flux $\alpha\in H_\pi^1(\mathbb{R})$ is given, then there exists a unique time periodic solution v^α of this problem (1.7)–(1.9) satisfying

$$\begin{aligned} &v^\alpha\in L_\pi^2(\mathbb{R};H_0^1(\Sigma)\cap H^2(\Sigma))\cap C_\pi(\mathbb{R};H_0^1(\Sigma)),\\ &(v^\alpha)'\in L_\pi^2(\mathbb{R};L^2(\Sigma)). \end{aligned}$$

Set

$$\begin{aligned} \boldsymbol{V}^\alpha(t,x) &= (v^\alpha(t,x),0) &\quad (n=2),\\ \boldsymbol{V}^\alpha(t,x) &= (v^\alpha(t,x),0,0) &\quad (n=3). \end{aligned}$$

Let us call $\boldsymbol{V}^\alpha$ "the time periodic Poiseuille flow".

2 Problem in a perturbed channel

Let Ω be a smooth and unbounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ and $\partial\Omega$ be the boundary of the domain Ω. A domain Ω is called a perturbed channel if Ω satisfies

$$\Omega \backslash B(0, R) = \omega \backslash B(0, R)(=: \omega_0),$$

where $B(0, R) = \{x \in \mathbb{R}^n; |x| < R\}$. ω_0 is a perturbed and bounded part, ω_L is channel parts. The boundary $\partial\Omega$ of Ω has connected components Γ_0, Γ_1, ..., Γ_J of C^∞-surface such that $\Gamma_1, \ldots, \Gamma_J$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and such that $\partial\Omega = \bigcup_{j=0}^{J} \Gamma_j$. Let us call the domain Ω "a perturbed channel".

In the domain Ω, we consider the nonstationary Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0, T) \times \Omega, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad (0, T) \times \Omega \tag{2.2}$$

with the boundary condition

$$\boldsymbol{u} = \boldsymbol{\beta} \qquad \text{on} \quad (0, T) \times \partial\Omega, \tag{2.3}$$

$$\boldsymbol{u} \to \boldsymbol{V}^\alpha \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \omega_L \tag{2.4}$$

and the time periodic condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \quad \text{in} \quad \Omega, \tag{2.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in Ω respectively, while $\nu > 0$ is the kinematic viscosity, $\boldsymbol{f} = \boldsymbol{f}(t, x)$ is the given external force and $\boldsymbol{\beta} = \boldsymbol{\beta}(t, x)$ is the given function on $(0, T) \times \partial\Omega$ with compact support. Since the solution $\boldsymbol{u}(t)$ satisfies $\operatorname{div} \boldsymbol{u}(t) = 0$ in Ω for a fixed $t \in (0, T)$, the given boundary data $\boldsymbol{\beta}(t)$ on $\partial\Omega$ is required to fulfill the compatibility condition which is called "General Outflow Condition" (GOC)

$$\int_{\partial\Omega} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = 0, \tag{2.6}$$

where $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$. The purpose is that if the given boundary date $\boldsymbol{\beta}$ satisfies (GOC), we will seek a solution of (2.1)-(2.5).

We introduce some function spaces. $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ is the set of all real smooth vector functions with compact support in Ω and $\operatorname{div} \boldsymbol{\varphi} = 0$. $\mathbb{L}_\sigma^2(\Omega)$ is the closure of $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual $\mathbb{L}^2(\Omega)$ norm. The $\mathbb{L}^2$ inner product and norm on Ω are denoted as $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_{2,\Omega}$ respectively. $\mathbb{H}_0^1(\Omega)$ and $\mathbb{H}_{0,\sigma}^1(\Omega)$ are the closures of $\mathbb{C}_0^\infty(\Omega)$ and $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual Dirichlet norm $\|\nabla \cdot\|_{2,\Omega}$, respectively. $\mathbb{H}_\sigma^1(\Omega)$ is the set of all $\mathbb{H}^1(\Omega)$ functions with $\operatorname{div} \boldsymbol{\varphi} = 0$. Let X be a Banach space. $C_\pi([0, T]; X)$ and $H_\pi^1((0, T); X)$ are the set of all the $C([0, T]; X)$ and $H^1((0, T); X)$ functions satisfying the time periodic condition $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ in X.

3 Result

Our definition of a time periodic weak solution of the Navier-Stokes equations (2.1), (2.2), (2.3), (2.4), (2.5) is as follows.

Definition 3.1 *A measurable function* $\boldsymbol{u} = \boldsymbol{u}(t,x)$ *on* $(0,T)\times\Omega$ *is called a time periodic weak solution of the Navier-Stokes equations* (2.1), (2.2), (2.3), (2.4), (2.5) *if* $\boldsymbol{u}$ *satisfies the following condition.*
(1) $\boldsymbol{v} := \boldsymbol{u} - \hat{\boldsymbol{V}}^{\alpha} - \boldsymbol{b} \in L^2((0,T);\mathbb{H}^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T);\mathbb{L}^2_\sigma(\Omega))$.
(2) $\boldsymbol{u}$ *satisfies* $\dfrac{d}{dt}(\boldsymbol{u},\boldsymbol{\varphi}) + \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi}) = {}_{(\mathbb{H}^1_{0,\sigma})'}\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{\mathbb{H}^1_{0,\sigma}} \quad (\boldsymbol{\varphi}\in\mathbb{H}^1_{0,\sigma}(\Omega))$.
(3) $\boldsymbol{v}(0) = \boldsymbol{v}(T) \in \mathbb{L}^2(\Omega)$,
where the function $\hat{\boldsymbol{V}}^{\alpha}$ *and* $\boldsymbol{b}$ *are to be such that*

$$\begin{aligned} \operatorname{div}\hat{\boldsymbol{V}}^{\alpha} &= 0 \quad \text{in} \quad \Omega \\ \hat{\boldsymbol{V}}^{\alpha} &= \boldsymbol{0} \quad \text{on} \quad \partial\Omega, \\ \hat{\boldsymbol{V}}^{\alpha} &= \boldsymbol{V}^{\alpha} \quad \text{in} \quad \omega_L, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}\boldsymbol{b} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{b} &= \boldsymbol{\beta} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

$\boldsymbol{V}^{\alpha}$ *is "the extended time periodic Poiseuille flow" and* $\boldsymbol{b}$ *is "the boundary extension".*

Before stating our result, we define a constant concerning the time periodic Poiseuille flow.

Definition 3.2 *We set*

$$\gamma^{\alpha}(t) = \sup_{\boldsymbol{\varphi}\in\mathbb{H}^1_{0,\sigma}(\omega)} \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{V}^{\alpha}(t))_{\omega}}{\|\nabla\boldsymbol{\varphi}\|^2_{2,\omega}} \quad (t\in[0,T]), \tag{3.1}$$

$$\hat{\gamma}^{\alpha} := \sup_{t\in[0,T]} \gamma^{\alpha}(t). \tag{3.2}$$

We have the following result.

Theorem 3.1 *(T. Kobayashi[13])*
Suppose that $\hat{\gamma}^{\alpha} < \nu$, $\boldsymbol{f} \in L^2((0,T);(\mathbb{H}^1_{0,\sigma}(\Omega))')$ *and* $\boldsymbol{\beta} = \boldsymbol{0}$. *Then there exists a time periodic weak solution.*

This result is not the problem of (GOC) because $\boldsymbol{\beta} = \boldsymbol{0}$. We need the following assumption.

Assumption 3.1 Ω *is a two dimensional symmetric domain with respect to the* x_1*-axis and all the inner boundaries* $\Gamma_j (1 \le j \le J)$ *intersect the* x_1*-axis.*

Theorem 3.2 *(T. Kobayashi[14])*
We assume that the domain Ω *satisfies* Assumption 3.1. *We suppose that* $\hat{\gamma}^{\alpha} < \nu$, $\boldsymbol{f} \in L^2((0,T);(\mathbb{H}^1_{0,\sigma}(\Omega))')$, $\boldsymbol{\beta} \in H^1_\pi((0,T);\mathbb{H}^{\frac{1}{2},S}(\partial\Omega))$ *with compact support,* (GOC) *and*

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\boldsymbol{n}\,d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\boldsymbol{n}\,d\sigma = 0 \quad \text{on} \quad [0,T].$$

Then there exists a time periodic weak solution of the Navier-Stokes equations.

We need an appropriate extension of the given boundary data $\boldsymbol{\beta}$.

Proposition 3.1 *We assume that a domain* Ω *satisfies* Assumption 3.1. *Suppose that* $\boldsymbol{\beta} \in H^1_\pi((0,T);\mathbb{H}^{\frac{1}{2},S}(\partial\Omega))$ *satisfies (GOC), the support of* $\boldsymbol{\beta}$ *is compact and*

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = 0 \quad \text{on} \quad [0,T].$$

Then for any $\varepsilon > 0$ *there exists an extension* $\boldsymbol{b}_\varepsilon \in H^1_\pi((0,T);\mathbb{H}^{1,S}_\sigma(\Omega))$ *of* $\boldsymbol{\beta}$ *such that* $\boldsymbol{b}_\varepsilon$ *has compact support and the inequality*

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{b}_\varepsilon(t))| < \varepsilon\|\nabla\boldsymbol{v}\|^2_{2,\Omega} \quad (\boldsymbol{v}\in\mathbb{H}^{1,S}_{0,\sigma}(\Omega), t\in[0,T]) \tag{3.3}$$

holds true.

The estimate (3.3) is "Leray's inequality". The estimate (3.3) is its symmetric version in an unbounded perturbed channel.

Remark 3.1 *In this paper, the domain* Ω *has two outlets. We can solve* K $(K \geq 3)$ *outlets problem. We consider a straight channel* ω_i $(i = 1,\cdots,K)$*, where* Σ_i *is a cross section of* ω_i *as* Section 1 *and the center line of* ω_i *may not be parallel to the* x_1*-axis. We assume that a given flux function* $\alpha_i \in H^1_\pi(\mathbb{R})$ $(i = 1,\cdots,K)$ *satisfies* $\sum_{i=1}^K \alpha_i(t) = 0$ $(t \in \mathbb{R})$*. For each* α_i*, we have the time periodic Poiseuille flow* $\boldsymbol{V}^\alpha_i$ *in* ω_i*. We assume that* Ω *has* K *outlets* ω_{0i} $(i = 1,\cdots,K)$ *where* ω_{0i} *is a semi-infinite channel with the cross section* Σ_i*. In the domain* Ω*, we consider a time periodic problem with the time periodic Poiseuille flow* $\boldsymbol{V}^\alpha_i$*. We define constant* $\hat{\gamma} = \max_{1\leq i\leq K}\{\hat{\gamma}^\alpha_i\}$ *as Definition* 3.2*. Suppose that* $\hat{\gamma} < \nu$*. Then there exists a time periodic weak solution in* Ω *with* K *outlets.*

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